The net evaluation is shown in the lower right corners of the corresponding cell

$u_i \backslash v_j$	23	27	30	50	
0	[22] 23	[8] 27	16 [14]	18 [32]	30
-10	12 [1]	[27] 17	[13] 20	51 [-11]	40
-18	22 [-17]	28 [-19]	[12] 12	[41] 32	53
	22	35	25	41	123

Now the net evaluation 32 is most positive and is at the cell (1, 4). Now considering the loop starting from (1, 4), by adding and subtracting θ to the cells, we observe that value for θ can be chosen from cells assigned by - θ in the loop so as not to get any negative allocation for any of the cells.

[22] 23	[8 - θ] 27	16 [14]	+θ 18 [32]	30
12 [1]	[27 +θ] 17	[13 - θ] 20	51 [- 11]	40
22 [-17]	28 [-19]	[12+θ] 12	[41 - θ] 32	53
22	35	25	41	123

Choosing θ = Min {41, 13, 8} = 8 which is allocated for the cell (1, 2) in the loop, thus (1, 4) should enter the basic and (1, 2) should leave from the basic. Thus we have the following improved basic feasible table

Table – 1

[22] 23	27	16	[8] 18	30
12	[35] 17	[5] 20	51	40
22	28	[20] 12	[33] 32	53
22	35	25	41	123

The new cost of the schedule is

$$Z = 22 \times 23 + 8 \times 18 + 35 \times 17 + 20 \times 15 + 20 \times 12 + 33 \times 32 = 2819 \text{ (improved).}$$

Second iteration

Basic variable	i	j	$c_{ij} = u_i + v_j$	Solution
x_{11}	1	1	$23 = 0 + v_1$	$v_1 = 23$
x_{14}	1	4	$18 = 0 + v_4$	$v_4 = 18$
x_{34}	3	4	$32 = u_3 + 18$	$u_3 = 14$
x_{33}	3	3	$12 = 14 + v_3$	$v_3 = -2$
x_{23}	2	3	$20 = u_2 - 2$	$u_2 = 22$
x_{22}	2	2	$17 = 22 + v_2$	$v_2 = -5$

Non-basic variable	i	j	$z_{ij} = u_i + v_j$	c_{ij}	Net evaluation $z_{ij} - c_{ij}$
x_{12}	1	2	$u_1 + v_2 = 0 - 5 = -5$	27	- 32
x_{13}	1	3	$u_1 + v_3 = 0 - 2 = -2$	16	-18
x_{21}	2	1	$u_2 + v_1 = 22 + 23 = 45$	12	33
x_{24}	2	4	$u_2 + v_4 = 22 + 18 = 40$	51	- 11
x_{31}	3	1	$u_3 + v_1 = 14 + 23 = 37$	22	15
x_{32}	3	2	$u_3 + v_2 = 14 - 5 = 9$	28	- 19

Non-basic cell (2, 1) is most positive. Now considering the loop starting from (2, 1), by adding and subtracting θ to the cells, we observe that value for θ can be chosen from cells assigned by - θ in the loop so as not to get any negative allocation for any of the cells.

$u_i \backslash v_j$	23	-5	-2	18	
0	[22 - θ] 23	27 [-32]	16 [-18]	[8 + θ] 18	30
22	[+ θ] 12 [33]	[35] 17	[5 - θ] 20	51 [-11]	40
14	22 [15]	28 [-19]	[20 + θ] 12	[33 - θ] 32	53
	22	35	25	41	123

Choosing θ = Min {22, 33, 5} = 5 which is allocated for the cell (2, 3) in the loop, thus (2, 1) should enter the basic and (2, 3) should leave from the basic. Thus we have the following improved basic feasible table

Table – 2

[17] 23	27	16	[13] 18	30
[5] 12	[35] 17	20	51	40
22	28	[25] 12	[28] 32	53
22	35	25	41	123

The cost of the new schedule is Z = 17×23+13×18+5×12+35×17+25×12+28×32 = 2476.

Third iteration

The multipliers and the corresponding net-evaluations are shown in the following table.

$u_i \backslash v_j$	23	28	-2	18	
0	[17] 23	27 [1]	16 [-18]	[13] 18	30
-11	[5] 12	[35] 17	20 [-33]	51 [-44]	40
14	22 [15]	28 [14]	[25] 12	[28] 32	53
	22	35	25	41	123

Non-basic cell (3, 1) is most positive. Now considering the loop starting from (3, 1), by adding and subtracting θ to the cells, we observe that value for θ can be chosen from cells assigned by - θ in the loop so as not to get any negative allocation for any of the cells.

	[17-θ] 23	27 [1]	16 [-18]	[13+θ] 18	30
	[5] 12	[35] 17	20 [-33]	51 [-44]	40
	+θ 22 [15]	28 [14]	[25] 12	[28-θ] 32	53
	22	35	25	41	123

Choosing θ = 17 which is allocated for the cell (1, 1) in the loop, we have the cell (3, 1) should enter the basic and (1, 1) should leave from the basic. Thus we have the following improved basic feasible table

Table – 3

23	27	16	[30] 18	30
[5] 12	[35] 17	20	51	40
[17] 22	28	[25] 12	[11] 32	53
22	35	25	41	123

The cost of the improved schedule is Z = 30×18+5×12+35×17+17×22+25×12+11×32 = 2221

Fourth iteration

The multipliers and the corresponding net-evaluations are shown in the following table.

$u_i \backslash v_j$	8	13	-2	18	
0	23 (-8)	27 (-14)	16 (-18)	(30) 18	30
4	(5) 12	(35) 17	20 (-18)	51 (-29)	40
14	(17) 22	28 (-1)	(25) 12	(11) 32	53
	22	35	25	41	123

The net evaluation is non-positive for each cell. Hence the optimal solution is reached. The optimal value of the schedule is Z = 2221.

Example 2. Stating with Vogel's method, find the optimal solution using MODI method for the following problem given in tabulated form. Also find the minimum cost for the schedule.

Source ↓ Destination →	1	2	3	4	Availability
1	10	2	20	11	15
2	12	7	9	20	25
3	4	14	16	18	10
Requirement	5	15	15	15	

Solution:

Step 1. The basic feasible solution to the given TP using Vogel's method is shown in the following table.

Table – 1

Source ↓ Destination →	1	2	3	4	Availability
1	10	(15) 2	20	11	15
2	12	7	(15) 9	(10) 20	25
3	(5) 4	14	16	(5) 18	10
Requirement	5	15	15	15	50

The cost of the corresponding schedule is $Z = 2\times15+9\times15+20\times10+4\times5+18\times5 = 475$.

Step 2. There are 5 cells for which the allocation is made and $m + n - 1 = 4 + 3 - 1 = 6$. Hence $m + n - 1 \neq 5$, the number of occupied cells. Thus, we produce a small positive assignment ε (≈ 0) to the cell (1, 1), so that the number of occupied cells is exactly equal to $m + n - 1$.

Source ↓ Destination →	1	2	3	4	Availability
1	10	[15] 2	20	11	15 + ε
2	12	[ε] 7	[15] 9	[10] 20	25
3	[5] 4	14	16	[5] 18	10
Requirement	5	15+ε	15	15	50

Step 3. For each occupied cell (corresponds to basic variables) in the current solution, computing multipliers u_i and v_j using the (u, v) equation $c_{ij} = u_i + v_j$, with $u_1 = 0$, we write u_i in the left of the corresponding rows and that of v_j on the top of the simplex table as:

	v_j	1	2	4	15	
u_i	Source ↓ Destination →	1	2	3	4	Availability
0	1	10	[15] 2	20	11	15
5	2	12	[ε] 7	[15] 9	[10] 20	25+ ε
3	3	[5] 4	14	16	[5] 18	10
	Requirement	5	15+ε	15	15	50

Step 4. For each unoccupied cells (corresponds to non-basic variables), finding the net evaluation $z_{ij} - c_{ij} = u_i + v_j - c_{ij}$ and entering these in the lower right corners of the corresponding cells.

	v_j	1	2	4	15	
u_i	Source ↓ Destination →	1	2	3	4	Availability
0	1	10 [-9]	[15] 2	20 [-16]	11 [4]	15
5	2	12 [-6]	[ε] 7	[15] 9	[10] 20	25+ ε
3	3	[5] 4	14 [-9]	16 [-9]	[5] 18	10
	Requirement	5	15+ε	15	15	50

Step 5. The net evaluation is not negative or zero for all non-basic variables. The most positive net evaluation is 4, which is at the cell (1, 4). Starting from the cell (1, 4) selected above, we get a loop as follows:

u_i	v_j / Source ↓ Destination	1 / 1	2 / 2	4 / 3	15 / 4	Availability
0	1	10 [-9]	[15-θ] 2	20 [-16]	[+θ] 11 [4]	**15**
5	2	12 [-6]	[ε+θ] 7	[15] 9	[10-θ] 20	**25+ ε**
3	3	[5] 4	14 [-9]	16 [-9]	[5] 18	**10**
	Requirement	**5**	**15+ε**	**15**	**15**	**50**

Adding and subtracting interchangeably the quantity θ = minimum of the allocations in the cells lies in the loop = 10 (except dummy allocation*), to and from the transition cells of the loop in such a way that the rim requirements remains satisfied, we get

u_i	v_j / Source ↓ Destination	1	2	3	4	Availability
	1	10	[5] 2	20	[10] 11	**15**
	2	12	[ε+ 10] 7	[15] 9	20	**25+ ε**
	3	[5] 4	14	16	[5] 18	**10**
	Requirement	**5**	**15+ε**	**15**	**15**	**50**

The cost of the new schedule is $Z = 2\times5+10\times11+10\times7+15\times9+5\times4+5\times18 = 435^{**}$ (improved).

Step 3. For each occupied cell (corresponds to basic variables) in the current solution, computing u_i and v_j using the uv-equation $c_{ij} = u_i + v_j$, with $u_1 = 0$ and then computing the net-evaluation of non-basic cells as above we get.

---

* Care should be taken while choosing the dummy cell, the degeneracy may occur while subtracting θ, the dummy cell in the loop should not include ε - θ.

** Here we take ε → 0.

	v_j	-3	2	4	11	
u_i	Source ↓ Destination →	1	2	3	4	Availability
0	1	10 [-13]	[5] 2	20 [-16]	[10] 11	15
5	2	12 [-10]	[ε+ 10] 7	[15] 9	20 [-5]	25+ ε
7	3	[5] 4	14 [-5]	16 [-5]	[5] 18	10
	Requirement	5	15+ε	15	15	50

The net-evaluations for the non-basic variables are all non-positive. Hence the optimal solution is reached. The cost of the schedule is Z = 435 is the optimal value of the TP and the optimal solution is:

'Transfer 5 units from source 1 to destination 2; 10 units from source 1 to destination 4; 10 units from source 2 to destination 2; 9 units from source 2 to destination 3; 5 units from source 3 to destination 1 and 5 units from source 3 to destination 4'.

Example 3: Solve the following transportation problem using Northwest-corner Rule for initial solution, whose cost is to be minimized:

	I	II	III	IV	Supply
A	13	11	15	20	2,000
B	17	14	12	13	6,000
C	18	18	15	12	7,000
Demand	3000	3000	4000	5000	

Solution: Using northwest-corner rule; the initial basic feasible table is

Table – 0

[2000] 13	11	15	10	2000
[1000] 17	[3000] 14	[2000] 12	13	6000
18	18	[2000] 15	[5000] 12	7000
3000	3000	4000	5000	15000

The cost corresponding to the schedule is

$Z = 13 \times 2000 + 17 \times 1000 + 14 \times 2000 + 12 \times 2000 + 15 \times 2000 + 12 \times 5000 = \underline{1,99,000}$

First iteration using MODI method:

Computing the multipliers and net evaluations we have.

$u_i \backslash v_j$	13	10	8	5	
0	[2000] 13	11 [-1]	15 [-7]	10 [-15]	2000
4	[1000-θ] 17	[3000] 14	[2000+θ] 12	13 [-4]	6000
7	[+θ] 18 [2]	18 [-1]	[2000-θ] 15	[5000] 12	7000
	3000	3000	4000	5000	15000

Net evaluations for all the non-basic variables are not all non-positive. The largest positive evaluation is 2 and is at the cell (3, 1). Hence the cell (3, 1) should enter the basis. Further the maximum possible value for θ is 1000 allocated for the cell (2, 1), hence (2, 1) should leave the basis. The improved feasible table is

Table – 1

[2000] 13	11	15	10	2000
17	[3000] 14	[3000] 12	13	6000
[1000] 18	18	[1000] 15	[5000] 12	7000
3000	3000	4000	5000	15000

The cost of the corresponding schedule is

$$Z = 2000 \times 13 + 1000 \times 18 + 3000 \times 14 + 3000 \times 12 + 1000 \times 15 + 5000 \times 12 = \underline{1,97,000}$$

Second iteration:

Computing multipliers and net evaluations we get

$u_i \backslash v_j$	13	12	10	7	
0	[2000-θ] 13	[+θ] 11 [1]	15 [-5]	10 [-3]	2000
2	17 [-2]	[3000-θ] 14	[3000+θ] 12	13 [-4]	6000
5	[1000+θ] 18	18 [-1]	[1000-θ] 15	[5000] 12	7000
	3000	3000	4000	5000	15000

Net evaluations for all the non-basic variables are not all non-positive. The largest positive evaluation is 1 and is at the cell (1, 2). Hence the cell (1, 2) should enter the basis. Further the maximum possible value for θ is 1000 allocated for the cell (3, 2), hence (3, 2) should leave the basis. The improved feasible table is

Table – 2

[1000] 13	[1000] 11	15	10	**2000**
17	[2000] 14	[4000] 12	13	**6000**
[2000] 18	18	15	5000 12	**7000**
3000	**3000**	**4000**	**5000**	**15000**

$$Z = 1000 \times 3 + 1000 \times 11 + 2000 \times 14 + 4000 \times 12 + 2000 \times 18 + 5000 \times 12 = \text{Rs. } \underline{1{,}96{,}000}$$

Third iteration:

Computing multipliers and net evaluations we get

$u_i \backslash v_j$	**13**	**11**	**9**	**7**	
0	[1000] 13	[1000] 11	15 [-6]	10 [-3]	**2000**
3	17 [-1]	[2000] 14	[4000] 12	13 [-3]	**6000**
5	[2000] 18	18 [-2]	15 [-1]	[5000] 12	**7000**
	3000	**3000**	**4000**	**5000**	**15000**

Net evaluations for all the non-basic cells are non-positive. Hence the optimal solution is reached and the minimum cost of transfer is Rs. 1,96,000. Further the optimal schedule;

Transfer 1000 units from *A* to I, 1000 units from *A* to II, 2000 units from *B* to II, 4000 units from *B* to III, 2000 units from *C* to I and 5000 units from *C* to IV. That is

$$x_{11} = 1000,\ x_{12} = 1000,\ x_{22} = 2000,\ x_{23} = 4000,\ x_{31} = 2000 \text{ and } x_{34} = 5000.$$

Example 4. A company manufacturing air coolers has two plants located at Bangalore and Bombay with a weekly capacity of 200 units and 100 units respectively. The company supplies air-coolers to its 4 showrooms situated at Mangalore, Bangalore, Delhi, and Goa which are have a demand of 75, 100, 100 and 25 units respectively. Due to the differences in local taxes, showroom charges, transportation cost and others, the profit differ. The profits (in Rs) are shown in the following table:

From	To			
	Mangalore	Bangalore	Delhi	Goa
Bangalore	90	90	100	100
Bombay	50	70	130	85

Plan the production program so as to maximize the profit. The company may have its production capacity at both plants partially or wholly unused.

Solution

Given problem in tabulated form is: Maximize Z so that

	Mangalore	Bangalore	Delhi	Goa	
Bangalore	90	90	100	100	**200**
Bombay	50	70	130	85	**100**
	75	**100**	**100**	**25**	**300**

Finding a initial feasible solution using northwest-corner rule: we get

Table – 0

[75] 90	[100] 90	[25] 100	100	**200**
50	70	[75] 130	[25] 85	**100**
75	**100**	**100**	**25**	**300**

Profit from this schedule is Z = 6750+9000+250+9750+2125 = 27875

Applying MODI method: $m + n - 1 = 2 + 4 - 1 = 5$ = number of occupied cells.

$u_i \backslash v_j$	**90**	**90**	**100**	**55**	
0	[75] 90	[100] 90	[25-θ] 100	+θ 100 [-45]	**200**
30	50 [70]	70 [50]	[75+θ] 130	[25-θ] 85	**100**
	75	**100**	**100**	**25**	**300**

Since the objective is to maximize, choosing the most negative evaluation – 45 assigned to the cell (1, 4), we have, the cell (1, 4) should enter the basis. Choosing θ = 25 we observe that two cells namely ((1, 3) and (2, 4) leave the basis. Thus allocating an allocation ε > 0 to one of the cell say (1, 3) so as to meet the condition $m + n - 1$ = number of occupied cells, and leaving the cell (2, 4) from the basis we get.

Table – 1

[75] 90	[100] 90	100	[25] 100	**200**
50	70	[100] 130	[ε] 85	**100+ε**
75	**100**	**100**	**25+ε**	**300**

Profit from the schedule is Z = 6750 + 9000 + 250 + 13000 = 31250.

$u_i \backslash v_j$	90	100	145	100	
0	[75] 90	[100] 90	100 [45]	[25] 100	200
-15	50 [25]	70 [15]	[100] 130	[ε] 85	100+ε
	75	100	100	25+ε	300

Since the net evaluation is non-negative for each of the nonbasic cells. The optimal solution is reached. The maximum profit is Rs. 31250. To achieve the profit; showroom at Mangalore should sell 75 coolers taking from the factory located at Bangalore, showroom at Bangalore should sell 100 coolers receiving from Bangalore, showroom located at Delhi should sell 100 coolers receiving from Bombay, and the showroom at Goa should sell 25 coolers buying from the factory located at Bangalore.

3.9 Unbalanced Transportation Problems

If the TP is not balanced, then we have introduce dummy sources or sinks to make it balance, which can be best explained with the help of following examples:

Example 5. A manufacturer must produce a product in sufficient quantity to meet contractual sales in next four months. The production capacity and unit cost production vary from month to month. The product produced in one month may be held for sale in later months but at an estimated storage cost of Rs. 1 per unit per month. No storage cost is incurred for goods sold in the same month in which they are produced. There is no opening inventory and none is desired at the end of four months. The necessary details are given in the following table:

Month	Contracted sales	Maximum production	Unit cost of production
1	20	40	14
2	30	50	16
3	50	30	15
4	40	50	17

How much should the manufacturer produce each month to minimize total cost?

Solution: Given data in the standard TP table is:

Month	1	2	3	4	
1	0+14	1+14	2+14	3+14	40
2	∞	0+16	1+16	2+16	50
3	∞	∞	0+15	1+15	30
4	∞	∞	∞	0+17	50
	20	30	50	40	

Note: In the above table we assign a cost ∞ to some cells*, which indicates the impossibility in supply of goods manufactured in the present month to the last month. Further the total cost of each month is the sum of cost of the month and the storage cost. Hence the table is not balanced.

The modified balanced table of the above problem by introducing a dummy column is:

14	15	16	17	0	**40**
∞	16	17	18	0	**50**
∞	∞	15	16	0	**30**
∞	∞	∞	17	0	**50**
20	**30**	**50**	**40**	**30**	**170**

Finding a initial basic feasible solution using northwest-corner rule:

Table – 0

[20] 14	[20] 15	16	17	0	**40**
∞	[10] 16	[40] 17	18	0	**50**
∞	∞	[10] 15	[20] 16	0	**30**
∞	∞	∞	[20] 17	[30] 0	**50**
20	**30**	**50**	**40**	**30**	**170**

There are 8 occupied basic cells and $m + n - 1 = 4 + 5 - 1 = 8$. Hence the table satisfy the condition $m + n - 1$ = not the basic cells.

The cost of the schedule is Z = 280+300+160+680+150+320+340+0 = Rs. 2230

* Some authors take it as M, where M is very large positive number.

MODI iteration:

First iteration:

$u_i \backslash v_j$	14	15	17	18	1	
0	[20] 14	[20] 15	16 [1]	17 [1]	0 [1]	40
1	∞ [-∞]	[10] 16	[40-θ] 17	18 [1]	+θ 0 [2]	50
-2	∞ [-∞]	∞ [-∞]	[10+θ] 15	[20-θ] 16	0 [-1]	30
-1	∞ [-∞]	∞ [-∞]	∞ [-∞]	[20+θ] 17	[30-θ] 0	50
	20	30	50	40	30	170

The largest positive net evaluation is 2 and is at the cell (2, 4). Hence the cell (2, 4) should enter the basis. Further the maximum possible value for θ is 20 allocated for the cell (3, 3), hence (2, 4) should enter the basis and (3, 3) leave the basis. The improved feasible table is

Table – 1

[20] 14	[20] 15	16	17	0	40
∞	[10] 16	[20] 17	18	[20] 0	50
∞	∞	[30] 15	16	0	30
∞	∞	∞	[40] 17	[10] 0	50
20	30	50	40	30	170

Cost of the new schedule is Z = 280+300+160+340+450+680 = 2210.

Second iteration:

$u_i \backslash v_j$	14	15	16	16	-1	
0	[20] 14	[20] 15	16 [0]	17 [-1]	0 [-1]	40
1	∞ [-ve]	[10] 16	[20] 17 [0]	18 [-1]	[20] 0	50
-1	∞ [-ve]	∞ [-ve]	[30] 15	16 [-1]	0 [-2]	30
1	∞ [-ve]	∞ [-ve]	∞ [-ve]	[40] 17	[10] 0	50
	20	30	50	40	30	170

The net evaluation is non-positive from each cell. Hence the optimal solution is reached. The optimal value is Z = 2210 and the optimal solution

(20) 14	(20) 15	16 [0]	17 [-1]	40
∞ [-ve]	(10) 16	(20) 17 [0]	18 [-1]	30
∞ [-ve]	∞ [-ve]	(30) 15	16 [-1]	30
∞ [-ve]	∞ [-ve]	∞ [-ve]	(40) 17	40
20	30	50	40	140

That is, 40 units should be produced in the first month and 20 should be delivered in that month, 30 units should be produced in the second month and 30 should be delivered in the second month, 30 units should be produced in the third month and 50 should be delivered in the third month, and, 40 should be produced in the fourth month and 40 should be delivered in the fourth month. Thus

Month	Production	Sales
1	40	20
2	30	30
3	30	50
4	40	40

Example 6. A company has 4 warehouses and 3 stores. The surplus in the warehouses, the requirements of the stores and costs (in Rs.) of transporting one unit of the commodity be transported so that the total transportation cost is a minimum? Obtain the initial program by applying the least-cost method:

Warehouse		Store		Supply
	1	2	3	
1	7	5	9	30
2	7	8	14	40
3	4	10	5	20
4	11	8	12	80
Requirement	30	40	100	170

Solution:

One of the initial feasible solutions, using least cost method is;

Table – 0

7	[30] 5	9	**30**
[10] 7	[10] 8	[20] 14	**40**
[20] 4	10	5	**20**
11	8	[80] 12	**80**
30	**40**	**100**	**170**

Cost of the schedule is Z = 150+ 70+80+280+80+960=1620

First iteration: The multipliers and the corresponding net evaluations are shown in the following table.

$u\backslash v$	4	5	11	
0	7 [-3]	[30] 5	9 [2]	**30**
3	[10+θ] 7	[10] 8	[20-θ] 14	**40**
0	[20-θ] 4	10 [-5]	+θ 5 [6]	**20**
-1	11 [-8]	8 [-4]	[80] 12	**80**
	30	**40**	**100**	**170**

Most positive is 6 at (3, 4). (3, 4) should enter the basic. By taking $\theta = 20$, we observe that the cells (3, 1) and (2, 3) leave the basic. The new solution is

Table – 1

7	[30] 5	9	**30**
[30] 7	[10] 8	14	**40**
4	10	[20] 5	**20**
11	8	[80] 12	**80**
30	**40**	**100**	**170**

Cost of the new schedule is Z = 150+210+80+100+960=1500

Further, Finding the multipliers and the corresponding net evaluations we get,

u\v	4	5	9	
0	7 [-3]	[30] 5	[ε] 9	30
3	[30] 7	[10] 8	14 [-2]	40
-4	4 [-4]	10 [-9]	[20] 5	20
3	11 [-4]	8 [0]	[80] 12	80
	30	40	100	170

Since net evaluation is non-positive for all nonbasic cells the optimum value is reached. The optimal value is Z = 1500.

Example 7. In the unbalanced transportation problem in the following table, where the numbers indicate the transportation cost, if a unit from a source is not shipped out (to any destinations), a storage cost is incurred at the rate of Rs. 5, Rs. 4 and Rs. 3 per unit for source 1, 2, and 3, respectively. If additionally all the supply at source 2 must be shipped out completely to make room for a new product, determine the optimum-shipping schedule.

1	2	1	20
3	4	5	40
2	3	3	30
30	20	20	

Solution: Given TP in tabular form is

Destination	1	2	3	4	
Source 1	1	2	1	5	20
Source 2	3	4	5	∞*	40
Source 3	2	3	3	3	30
	30	20	20	20	90

Destination - 4 denote the undelivered goods. Finding initial basic feasible solution using northwest – corner rule we get;

Table – 0

[20] 1	2	1	5	20
[10] 3	[20] 4	[10] 5	∞	40
2	3	[10] 3	[20] 3	30
30	20	20	20	90

* here we introduce ∞ to make the room available for the new product.

Cost of the schedule is Z = 20+30+80+50+30+60 = 270.

Applying MODI-method: Initial iteration is

u\v	1	2	3	3	
0	[20-θ] 1	2 [0]	+θ 1 [2]	5 [-2]	20
2	[10+θ] 3	[20] 4	[10-θ] 5	∞ -ve	40
0	2 [-1]	3 [-1]	[10] 3	[20] 3	30
	30	20	20	20	90

The cell (1,3) is most positive, Forming a loop from (1, 3) we have θ = 10. Thus (1, 3) should leave the basic and (1, 3) should enter the basic. Improved schedule is given by

[10] 1	2	[10] 1	5	20
[20] 3	[20] 4	5	∞	40
2	3	[10] 3	[20] 3	30
30	20	20	20	90

The cost of the corresponding schedule is Z = 10+10+60+80+30+60 = 250.

Further the net evaluation for this schedule is

u\v	1	2	1	1	
0	[10-θ] 1	2 [0]	[10+θ] 1	5 [-4]	20
2	[20+θ] 3	[20-θ] 4	5 [-2]	∞ [-ve]	40
2	2 [1]	+θ 3 [1]	[10-θ] 3	[20] 3	30
	30	20		20	90

Second iteration: The net evaluation for the cells (3,1) and (3,2) are most positive. Thus (3, 2) should leave the basic (choice is arbitrary). The value of θ is = 10. Hence (3, 2) should enter the basic and both (1, 1) and (3, 3) should leave the basic. The improved schedule is

1	2	[20] 1	5	20
[30] 3	[10] 4	5	∞	40
2	[10] 3	3	[20] 3	30
30	20	20	20	90

The cost of the new schedule is Z = 20+60+80+20+60 = 240.

Finding net evaluation, by introducing a new basic cell say (1,4) (since $m + n - 1 \neq$ number of occupied cells),

	1	2	1	2	
0	1 [0]	[ε] 2	[20] 1	5 [-3]	**20**
2	[30] 3	[10] 4	5 [-2]	∞ [-ve]	**40**
1	2 [0]	[10] 3	3 [-1]	[20] 3	**30**
	30	**20**	**20**	**20**	**90**

The net evaluation for each nonbasic cell is non-positive. Hence the optimal solution is reached. The optimal value is 240. The optimal schedule is

	1	2	3
1	-	20	-
2	30	10	-
3	-	10	-

Example 8. Three refineries with daily capacities 6, 5 and 6 million gallons respectively, supply three distribution areas with daily demands of 4, 8 and 7 million gallons, respectively. Gasoline is transported to the three distribution areas through a network of pipelines. The transportation cost is Rs. 10 per 1000 gallons per pipeline mile. The following table gives the mileage between the refineries and the distribution areas. Refinery 1 is not connected to distribution area 3. Solve the TP with the condition that the distribution area 1 must receive all its demand and any shortage at areas 2 and 3 will incur a penalty of Rs. 5 per gallon.

		Distribution Area		
		1	**2**	**3**
	1	120	180	--
Plant	**2**	300	100	80
	3	200	250	120

Solution: The given TP in tabular form is

0.01×120	0.01×180	∞	**60,00,000**
0.01×300	0.01×100	0.01×80	**50,00,000**
0.01×200	0.01×250	0.01×120	**60,00,000**
∞	5	5	**20,00,000**
40,00,000	**80,00,000**	**70,00,000**	**90,00,000**

Finding a initial basic feasible schedule using northwest-corner rule we get

Table – 0

40,00,000 1.2	20,00,000 1.8	∞	60,00,000
3	50,00,000 1	0.8	50,00,000
2	10,00,000 2.5	50,00,000 1.2	60,00,000
∞	5	20,00,000 5	20,00,000
40,00,000	80,00,000	70,00,000	90,00,000

Cost of the schedule is

Z = 48,00,000+36,00,000+50,00,000+25,00,000+60,00,000+1,00,00,000 =3,19,00,000

Applying MODI method: First iteration is

u\v	1.2	1.8	0.5	
0	40,00,000 1.2	20,00,000 1.8	∞ -ve	60,00,000
-0.8	3 -2.6	50,00,000 1	0.8 0.38	50,00,000
0.7	2 -0.1	10,00,000-θ 2.5	50,00,000+θ 1.2	60,00,000
4.5	∞ -ve	+θ 5 1.3	20,00,000-θ 5	20,00,000
	40,00,000	80,00,000	70,00,000	90,00,000

The net evaluation to the cell (4, 2) is most positive. Hence (4, 2) should leave the basic. Now forming a loop starting from (4, 2) we get θ = 10,00,000. Adding and subtracting θ from and to the cells so as to meet the rim requirements we get the following improved schedule.

40,00,000 1.2	20,00,000 1.8	∞	60,00,000
3	50,00,000 1	0.8	50,00,000
2	2.5	40,00,000θ 1.2	60,00,000
∞	10,00,000 5	10,00,000 5	20,00,000
40,00,000	80,00,000	70,00,000	90,00,000

Cost of the schedule is

Z = 48,00,000+36,00,000+50,00,000+48,00,000+50,00,000+50,00,000 =2,82,00,000

Finding multipliers and net evaluation corresponding to this schedule we get,

u\v	1.2	1.8	1.8	
0	40,00,000 1.2	20,00,000 1.8	∞ -ve	60,00,000
-0.8	3 -2.6	50,00,000-θ 1	+θ 0.8 0.2	50,00,000
-0.6	2 -1.4	2.5 -1.3	40,00,0000 1.2	60,00,000
3.2	∞ -ve	10,00,000+θ 5	10,00,000-θ 5	20,00,000
	40,00,000	80,00,000	70,00,000	90,00,000

The net evaluation is most positive for the cell (2, 3). Hence forming a loop from (2,3) we get $\theta = 10,00,000$. The improved schedule is

40,00,000 1.2	20,00,000 1.8		∞	60,00,000
3	40,00,000 1	10,00,000	0.8	50,00,000
2	2.5	40,00,0000	1.2	60,00,000
∞	20,00,000 5		5	20,00,000
40,00,000	80,00,000	70,00,000		90,00,000

Cost of the new schedule is

$Z = 48,00,000+36,00,000+40,00,000+8,00,000+48,00,000+1,00,00,000 = 2,80,00,000$

Net evaluation for this schedule is

u\v	1.2	1.8		1.0	
0	40,00,000 1.2	20,00,000 1.8		∞ -ve	60,00,000
-0.2	3 -2	40,00,000 1	10,00,000	0.8	50,00,000
0.2	2 0.6	2.5 -0.5	40,00,0000	1.2	60,00,000
3.2	∞ -ve	20,00,000 5		5 -0.8	20,00,000
	40,00,000	80,00,000		70,00,000	90,00,000

Thus net evaluation for every nonbasic cells is zero. Hence optimal solution is reached. The optimal value is Rs. 2,80,00,000 and the optimal solution is

		Distribution Area		
		1	2	3
	1	40,00,000	20,00,000	–
Plant	2	0	40,00,000	10,00,000
	3	0	0	40,00,000

Example 9. Three orchards supply crates of oranges to four retailers. The daily demand amounts at the four retailers are 150, 150, 400 and 100 crates, respectively. Supply at the three orchards is dictated by available regular labor and are estimated at 150, 200 and 250 crates daily. However, both orchards 1 and 2 have indicated that they could supply more crates, if necessary, by using overtime labor with normal wages. Orchard 3 does not offer this option. The transportation costs per crates from the orchards to the retailer are given in the following table. Formulate the problem as a transportation model and hence using MODI method estimate the number of crates should orchards 1 and 2 supply using overtime labor in order to minimize the cost. What can you say if the overtime laborers wage is double the normal wage?

		Retailer			
		1	2	3	4
Orchard	1	1	2	3	2
	2	2	4	1	2
	3	1	3	5	3

Solution: Given problem in tabular form after introducing a dummy row to meet the rim condition is given below:

	1	2	3	4	
1	1	2	3	2	150
2	2	4	1	2	200
3	1	3	5	3	250
I	0	0	0	0	200
	150	150	400	100	800

Initial solution using northwest corner rule is:

Table – 0

150 1	2	3	2	150
2	150 4	50 1	2	200
1	3	250 5	3	250
0	0	100 0	100 0	200
150	150	400	100	800

Cost of the schedule is Z = 150 + 600 + 50 + 1250 = 2,050.

Applying MODI method to improve the solution we get

Initial iteration:

u\v	1	2	-1	-1	
0	[150-θ] 1	[ε+θ] 2	3 (-4)	2 (-3)	150
2	2 (1)	[150-θ] 4	[50+θ] 1	2 (-1)	200
6	[+θ] 1 (6)	3 (5)	[250-θ] 5	3 (2)	250
1	0 (2)	0 (3)	[100] 0	[100] 0	200
	150	150	400	100	800

The net evaluation is most positive for the cell (3, 1). Hence forming a loop starting from (3, 1) we get θ = 150. The improved schedule is

Table – 1

1	[150] 2	3	2	150
2	4	[200] 1	2	200
[150] 1	3	[100] 5	3	250
0	0	[100] 0	[100] 0	200
150	150	400	100	800

Cost of the new schedule is Z = 300 + 200 + 150 + 500 = Rs. 1,050.
Finding multipliers and the corresponding net evaluations for the new table we get.

u\v	-2	2	2	2	
0	1 (-3)	[150] 2	3 (-1)	[ε] 2	150
-1	2 (-5)	4 -3	[200] 1	2 (-1)	200
3	[150] 1	3 (2)	[100-θ] 5	[+θ] 3 (2)	250
-2	0 (-4)	0 (0)	[100+θ] 0	[100-θ] 0	200
	150	150	400	100	800

The net evaluation is most positive for the cell (3, 1). Hence forming a loop starting from (3, 1) we get θ = 100. The improved schedule is

Table – 2

1	(150) 2	3	(ε) 2	150
2	4	(200) 1	2	200
(150) 1	3	5	(100) 3	250
0	0	(200) 0	0	200
150	150	400	100	800

Cost of the new schedule is Z = 300 + 200 + 150 + 300 = Rs. 950.

Finding multipliers and the corresponding net evaluations for the new table we get

u\v	0	2	3	2	
0	1 [-1]	(150) 2	(ε) 3	(ε) 2	150
-2	2 [-4]	4 [-4]	(200) 1	2 [-2]	200
1	(150) 1	3 [0]	5 [-1]	(100) 3	250
-3	0 [-3]	0 [-1]	(200) 0	0 [-1]	200
	150	150	400	100	800

Net evaluation is non-positive for all the cells hence an optimal solution is reached. The minimum cost of the schedule is Z = 950. Further we observe that the detailer 3 will not receive completely the desired crates of oranges as a dummy orchard supplies 200 units. Physically this orchard is absent. Thus these 200 units can be supplied to the detailer 3 by the orchard 2 at a minimum cost of Rs. 200. The total cost of the new schedule by meeting all the demands of the retailer is now Rs. 950 + Rs 200 = Rs. 1,150. We show now this schedule is again optimal by applying MODI method again to the new schedule as:

u\v	0	2	3	2	
0	1 [-1]	(150) 2 [0]	(ε) 3 [0]	(ε) 2 [0]	150
-2	2 [-4]	4 [-4]	(200) 1 [0]	2 [-2]	400
1	(150) 1 [0]	3 [0]	5 [-1]	(100) 3 [0]	250
	150	150	400	100	800

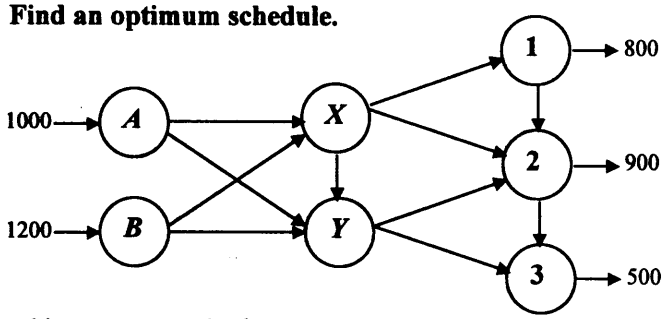
The net evaluation for all the cells is non-positive. Hence the conclusion is valid. Thus the orchard 2 only should supply 200 more crates of oranges to the detailer 3 by using overtime labor. Finally if the labour cost is double the normal rate for overtime, then also the solution is same as orchards manufacturing cost of 200 crates is Rs. 2 × 200 = 400 and that of orchard 1 is Rs. 3 × 200 = Rs. 600 (observe?).

3.10 Transshipment Model

The transshipment model recognizes that in real life it may be cheaper to ship through intermediate or transient nodes before reaching the final destination. This concept is more general than the one advanced by the regular transportation model, where direct shipments only are allowed between a source and a destination.

A transshipment model can be converted into a regular transportation model using the idea of buffer.

Example 10. Two automobile plants, *A* and *B* are linked to three dealers 1, 2 and 3, by way of two transit centers, *X* and *Y* according to the network shown in the following figure. The supply amount at plant *A* and *B* are 1000 and 2000 cars, and the demand amounts at dealers 1, 2, and 3, are 800, 900, and 300 cars respectively. The shipping cost per car between pairs of nodes is shown on the connecting links of the network. Find an optimum schedule.



Solution: Transshipment occurs in the network in the above figure because the entire supply amount of 2200 (= 1000 + 1200) cars at nodes *A*, and *B* could conceivably pass through any node of the network before ultimately reaching their destinations at nodes 1, 2, and 3.

In this regard, the nodes of the network with both input and output arcs X, Y, 1, 2 acts as both sources and destinations, and are referred as transshipment nodes. The remaining nodes are either pure supply nodes (i.e. A, B) or pure demand nodes (i.e. 3). The transshipment model can be converted into a regular transportation model with six sources X, Y, A, B, 1, and 2 and five destinations X, Y, 1, 2 and 3. The amounts of supply and demand at the different nodes are computed as

Supply at a pure supply node = Original supply

Supply at a transshipment node = Original supply + buffer

Demand at a pure demand node = Original demand

Demand at a transshipment node = Original demand + buffer.

The buffer amount should be sufficiently large to allow all the original supply (or demand) units to pass through any of the transshipment nodes. Let K be the desired buffer amount, then

K = Total supply (or demand) = 1000 + 1200 (or 800 + 900 + 500) = 2200

The equivalent regular transportation model is

	X	Y	1	2	3	
A	3	4	–	–	–	1000

B	2	5	--	--	--	1200
X	0	7	8	6	--	K
Y	--	0	--	4	9	K
1	--	--	0	5	--	K
2	--	--	--	0	3	K
	K	K	800+K	900+K	500	

The initial basic feasible solution starting with northwest corner rule we get, where M is very large*.

u\v	3	4	5	10	13	
0	[1000] 3	[ε] 4	M [-ve]	M [-ve]	M [-ve]	1000
-1	[1200] 2	5 [-2]	M [-ve]	M [-ve]	M [-ve]	1200
3	0 [6]	[2200] 7	[ε] 8	6 [7]	M [-ve]	2200
M-5	M [-2]	0 [M-9]	[2200-θ] M	+θ 4 [M+1]	9 [M-1]	2200
-5	M [-ve]	M [-ve]	[800+θ] 0	[1400-θ] 5	M [-ve]	2200
-10	M [-ve]	M [-ve]	M [-ve]	[1700] 0	[500] 3	2200
	2200	2200	3000	3100	500	

Choosing θ = 1400, we get second iteration as

u\v	3	4	5	9-M	12-M	
0	[1000] 3	[ε] 4	M [-ve]	M [-ve]	M [-ve]	1000
-1	[1200] 2	5 [-2]	M [-ve]	M [-ve]	M [-ve]	1200
3	0 [6]	[2200-θ] 7	[ε+θ] 8	6 [-ve]	M [-ve]	2200
M-5	M [-2]	+θ 0 [M-1]	[800-θ] M	[1400] 4	9 [-2]	2200
-5	M [-ve]	M [-ve]	[2200] 0	5 [-ve]	M [-ve]	2200
M-9	M [-12]	M [-13]	M [-14]	[1700] 0	[500] 3	2200
	2200	2200	3000	3100	500	

* here we take M instead of ∞, because we cannot choose multipliers (observe). Also 0 indicates stock at the junction corresponding to the node.

Choosing θ = 800, we have the next iteration:

u\v	3	4	5	8	11	
0	[1000-θ] 3	[ε+θ] 4	M (-ve)	M (-ve)	M (-ve)	1000
-1	[1200] 2	5 (-2)	M (-ve)	M (-ve)	M (-ve)	1200
3	[+θ] 0 (6)	[1400-θ] 7	[800] 8	6 (5)	M (-ve)	2200
-4	M (-ve)	[800] 0	M (-ve)	[1400] 4	9 (-2)	2200
-5	M (-ve)	M (-ve)	[2200] 0	5 (-2)	M (-ve)	2200
-8	M (-ve)	M (-ve)	M (-ve)	[1700] 0	[500] 3	2200
	2200	2200	3000	3100	500	

Choosing θ = 1000, we get the next iteration:

u\v	-3	4	5	8	11	
0	3 (-ve)	[1000] 4	M (-ve)	M (-ve)	M (-ve)	1000
5	[1200] 2	5 (4)	M (-ve)	M (-ve)	M (-ve)	1200
3	[1000] 0	[400-θ] 7	[800] 8	[+θ] 6 (5)	M (-ve)	2200
-4	M (-ve)	[800+θ] 0	M (-ve)	[1400-θ] 4	9 (-2)	2200
-5	M (-ve)	M (-ve)	[2200] 0	5 (-2)	M (-ve)	2200
-8	M (-ve)	M (-ve)	M (-ve)	[1700] 0	[500] 3	2200
	2200	2200	3000	3100	500	

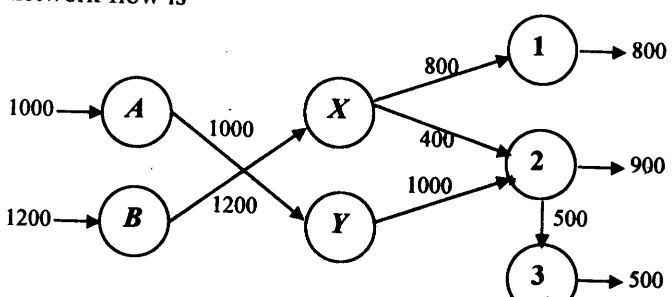
Choosing θ = 400, we have the next iteration:

u\v	2	4	10	8	11	
0	3 (-1)	[1000] 4	M (-ve)	M (-ve)	M (-ve)	1000
0	[1200] 2	5 (-1)	M (-ve)	M (-ve)	M (-ve)	1200
-2	[1000] 0	7 (-5)	[800] 8	[400] 6	M (-ve)	2200
-4	M (-ve)	[1200] 0	M (-ve)	[1000] 4	9 (-2)	2200
-10	M (-ve)	M (-ve)	[2200] 0	5 (-7)	M (-ve)	2200
-8	M (-ve)	M (-ve)	M (-ve)	[1700] 0	[500] 3	2200
	2200	2200	3000	3100	500	

Since the net evaluation is non-positive for each cell, an optimal solution is reached. The optimal solution is

	X	Y	1	2	3
A	3	1000 4	--	--	--
B	1200 2	5	--	--	--
X	1000 0	7	800 8	400 6	--
Y	--	1200 0	--	1000 4	9
1	--	--	2200 0	5	--
2	--	--	--	1700 0	500 3

That is optimal network flow is



Optimal value $Z = 1000 \times 4 + 1200 \times 2 + 800 \times 8 + 400 \times 6 + 1000 \times 4 + 500 \times 3 = 20{,}700$.
Stock at X is 0, Stock at Y is 0, Stock at 1 is 0, Stock at 2 is 500, and stock at 3 is 0.

EXERCISES

1. Three electric power plants with capacities of 25, 40 and 30 million kWh. Supply electricity to three cities, the maximum demands at three cities are estimated at 30, 35 and 25 million kWh. The price per million kWh at the three cities are given in table:

		City 1	City 2	City 3
Plant	1	Rs. 600	Rs. 700	Rs. 400
	2	Rs. 320	Rs. 300	Rs. 350
	3	Rs. 500	Rs. 480	Rs. 450

During the month of August, there is a 20% increase in demand at each of the three cities, which can be met by purchasing the electricity from another network at a premium rate of Rs. 1000 per million kWh. The network is not linked to city 3, however. The utility company wishes to determine the most economical plan for the distribution and the purchase of additional energy. Determine the cost of the additional power purchased by each of the three cities.

2. Solve the problem number 1 by assuming that there is a 10% power transmission loss through the network.

3. Three refineries with daily capacities of 6, 5 and 8 million gallons, respectively, supply three distribution areas with daily demands of 4, 8, and 7 million gallons respectively.

Gasoline is transported to the three distribution areas through a network of pipelines. The transportation cost is Rs. 10 per 1000 gallons per pipeline mile. The following table gives the mileage between the refineries and the distribution areas. Refinery 1 is not connected to distribution area 3.

		Distribution area		
		1	2	3
	1	120	180	---
Plant	2	300	100	80
	3	200	250	120

Determine the optimum-shipping schedule in the network.

4. Automobiles are shipped from three distribution centers to five dealers. The shipped cost is based on the mileage between the sources and the destination, and is independent of whether the truck makes the trip with partial or full loads. The following table summarizes the mileage between the distribution centers and the dealers together with the monthly supply and demand figures given in number of automobiles. A full truckload includes 18 automobiles. The transportation cost for truck mail is Rs. 25.

		Dealer					
		1	2	3	4	5	Supply
	1	100	150	200	140	35	**400**
Center	2	50	70	60	65	80	**200**
	3	40	90	100	150	130	**150**
Demand		**100**	**200**	**150**	**160**	**140**	

Determine the optimal shipping schedule.

5. The demands of essential commodities for the next four months are 400, 300, 420 and 380 tones respectively. The supply capacities for the same months are 500, 600, 200 and 300 tones. The purchase price per tone varies from month to month is estimated at Rs. 100, Rs. 140, Rs. 120 and Rs. 150 respectively. Because the item is essential commodities, a current month's supply must be consumed within three months (including the current month). The storage cost per tone per month is Rs. 3. The nature of the item does not allow returning back. Determine the optimum delivery schedule for the item over the next four months.

6. Using northwest-corner rule, determine the optimal solution to the following:

(i)

0	2	1	**6**
2	1	5	**9**
2	4	3	**5**
5	**5**	**10**	

(ii)

0	4	2	**8**
2	3	4	**5**
1	2	0	**6**
7	**6**	**6**	

(iii)

-	3	5	**4**
7	4	9	**7**
1	8	6	**19**
5	**6**	**19**	

(iv)

10	2	20	11	**15**
12	7	9	20	**25**
4	14	16	18	**10**
5	**15**	**15**	**15**	

7. In the unbalance transportation problem given in the following table, if a unit from a source is not shipped out (to any of the destinations), a storage cost is incurred at the rate of Rs. 5, Rs. 4 and Rs. 3 per unit for sources, 1, 2 and 3 respectively. If additionally all the supply at source 2 must be shipped out completely to make room for a new product, determine the optimum-shipping schedule.

1	2	1	**20**
3	4	5	**40**
2	3	3	**30**
30	**20**	**20**	

8. A departmental store wishes to stock the following quantities of a popular product in three types of container.

Container Type	1	2	3
Quantity	170	200	180

Tenders are submitted by four dealers who undertake to supply not more than the quantities shown below.

Dealer	1	2	3	4
Quantity	150	160	110	130

The store estimates that profit per unit bill vary with the dealer as shown below:

		Dealer			
		1	2	3	4
Container type	1	8	9	6	3
	2	6	11	5	10
	3	3	8	7	9

9. A company has received a contract to supply gravel for three new construction projects located in towns *A*, *B* and *C*. Construction engineers have estimated the required amounts of gravel which will be needed at these construction projects:

Project location	Weekly requirement (truck loads)
A	72
B	102
C	41

The company has 3 gravel pits located in town *W*, *X* and *Y*. The gravel required by the construction projects can be supplied by three pits. The amount of gravel, which can be supplied by each pit, is as follows:

Plant	*W*	*X*	*Y*
Amount available (truck loads)	76	82	77

The company has computed the delivery cost from each pit to each project site. These costs are shown in the following table.

		Project Location		
		A	*B*	*C*
Pit	*W*	Rs. 4	Rs. 8	Rs. 8
	X	Rs. 16	Rs. 24	Rs. 16
	Y	Rs. 8	Rs. 16	Rs. 24

Schedule the shipment from each pit to each project in such a manner so as to minimize the total transportation cost within the constraints imposed by pit capacities and project requirements. Find the minimum cost. Is the solution unique? if it is not, find alternative schedule.

10. A company has three warehouses in cities *A*, *B* and *C*. These warehouses have the following quantities of the product in stock:

Distributorships:	*A*	*B*	*C*
Capacity:	100 units	80 units	80 units

The four customers have the demands as follows:

(1) 60 units, (2) 120 units, (3) 50 units (4) 40 units.

If cost is 20 paise per km. to transport this product from a distributor to customer. The distance between warehouses and customers can be obtained from a map or suitable route tables as follows:

		Customer			
		1	2	3	4
	A	270	230	310	600
Warehouse	B	100	450	400	320
	C	300	540	350	570

Which warehouse should deliver how much product to which customer so that the total transportation cost becomes a minimum?

11. A company produces a small component for an industrial product and distributes it to five wholesalers at a fixed delivered price of Rs. 2.50 per unit. Sales forecasts indicate that monthly deliveries will be 3,000, 3,000, 10,000, 5,000 and 4,000 units to wholesalers 1, 2, 3, 4, and 5 respectively. The monthly production capacities are 5,000, 10,000 and 12,5000 at plants 1, 2 and 3 respectively. The direct costs of production of each unit are Rs. 1.00, Rs. 0.90 and Rs. 0.80at places 1, 2, and 3 respectively. The transportation costs of shipping a unit from a plant to a wholesaler are given below:

		Wholesaler				
		1	2	3	4	5
Plant	1	0.06	0.07	0.10	0.15	0.15
	2	0.80	0.06	0.09	0.12	0.14
	3	0.10	0.09	0.08	0.10	0.15

Find how many components each plant supplies to each wholesaler in order to maximize its profit.

12. In a transportation problem in the following table, the total demand exceeds the total supply. Suppose that the penalty costs per unit of unsatisfied demand are Rs. 5, Rs. 3 and Rs. 2 for destinations 1, 2, and 3, respectively. Determine the optimum solution.

5	1	7	**10**
6	4	6	**80**
3	2	5	**15**
75	**20**	**50**	

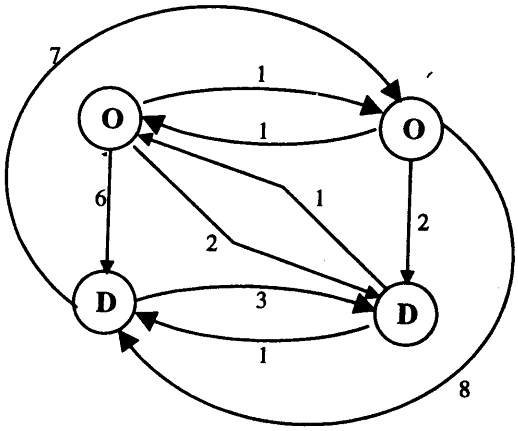
13. Three refineries with daily capacities 6, 5 and 8 million gallons respectively, supply three distribution areas with daily demands of 4, 8 and 7 million gallons, respectively. Gasoline is transported to the three distribution areas through a network of pipelines. The transportation cost is Rs. 10 per 1000 gallons per pipeline mile. The following table gives the mileage between the refineries and the distribution areas. Refinery 1 is not connected to distribution area 3. Solve the TP using MODI method so as to minimize the transportation cost.

14. The National Forest Service is receiving four bids for logging at three forests in Karnataka. The three locations include 10,000, 20,000 and 30,000 acres. A single bidder can bid for at most 50% of the total acreage available. The bids (in Rs.) per acre at the three locations are given in the following table.

		Location		
		1	2	3
	1	520	210	570
Bidder	2	--	510	495
	3	650	--	240
	4	180	430	710

Find an optimum schedule for the above problem using MODI method so that the total bidding revenue for the National Forest Service is Maximum.

15. There are two origins and two destinations in a transportauon problem. The origin availabilities are 4, 5 and requirements at the destinations are 3, 6. The transportation costs are indicated in the following network. Solve the problem.



16. A firm having two sources S_1 and S_2 wishes to ship its products to two destination D_1 and D_2. The number of units available at S_1 and S_2 are 5 and 25 respectively and the product demand at D_1 and D_2 are 20 and 10 units respectively. The firm, instead of shipping directly from sources to destinations decides to investigate the possibility of trans-shipment. The unit transportation costs (in rupees) are given in the following table. Find the optimal shipping schedule:

		Source		Destination		Availability
		S_1	S_2	D_1	D_2	
Source	S_1	0	2	3	4	5
	S_2	2	0	2	4	25
Destination	D_1	3	2	0	1	--
	D_2	4	4	1	0	--
	Demand	--	--	20	10	

Answers: 2. City 1 will purchase 22.5 million kWh from the network at a cost of 22,500.

3. Area 1 $\rightarrow$ 4 million from refinery 1, Area 2 $\rightarrow$ 2 million from 1, 5 million from 2, 1 million from 3, Area 3 $\rightarrow$ 7 from 3. Total cost 243,000.

5. Produce 500 in period 1, with 100 units carried over to period 2; produce 600 units in period 2, with 200 units carried over to period 3 and 180 units carried over to period 4; produce 200 in period 3; and produce 200 in period 4. Total cost 190,040

6. (i) 33, $\begin{bmatrix} 1 & 0 & 5 \\ 2 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ (iv) 435, $\begin{bmatrix} 0 & 5 & 0 & 10 \\ 0 & 10 & 15 & 0 \\ 5 & 0 & 0 & 5 \end{bmatrix}$, 12. 515, $\begin{bmatrix} 0 & 10 & 0 \\ 20 & 10 & 50 \\ 5 & 10 & 0 \end{bmatrix}$, shortage in 1 is 40.

15. Transport 4 units from O_1 to D_2. 5 units from O_2 to D_2, then 3 units from D_2 to D_1.

16. Transport 25 units fro S_2 to D_1. It increases the capacity of S_2 to 55 units including 30 units as buffer stock. From S_1, transport 5 units to D_2 , and out of 25 units available at D_1, transport 5 units to D_2.

3.11 Assignment Problems

Introduction: The assignment problem is a special case of the transportation problem. In assignment problem the objective is to assign a number of origins/workers to equal* number of destinations/jobs at a minimum cost or maximum profit. The ij^{th} element c_{ij} of the effectiveness matrix represents the cost or income of assigning worker i to job j.

3.12 Formulation:

Mathematical model for an assignment problem is as follows:

$$\text{Optimize } z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{n} x_{ij} = 1$$

and $x_{ij} \in \{0, 1\}$, for all $i, j \in \{1, 2, 3, \ldots, n\}$

Or in tabular form we write it as

$$\begin{array}{c} \text{worker} \\ \downarrow \\ 1 \\ 2 \\ \cdot \\ \cdot \\ n \end{array} \quad \begin{array}{c} \leftarrow \text{job} \rightarrow \\ \begin{array}{ccccc} 1 & 2 & \cdots & \cdots & n \end{array} \\ \begin{bmatrix} c_{11} & c_{12} & \cdots & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & \cdots & c_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{n1} & c_{n2} & \cdot & \cdot & c_{nn} \end{bmatrix} \end{array}$$

The basic concept on optimal solution of the assignment model is that the solution remains unchanged if a constant is added to or subtracted from any row or columns of the cost matrix. In fact, if p_i and q_j are constants subtracted from row i and column j, then the cost c_{ij} changes to $c_{ij}' = c_{ij} - p_i - q_j$ and the corresponding objective function is

$$Z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}' x_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} \left(c_{ij} - p_i - q_j\right) x_{ij}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_{ij} - \sum_{i=1}^{n}\sum_{j=1}^{n} p_i x_{ij} - \sum_{i=1}^{n}\sum_{j=1}^{n} q_j x_{ij}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_{ij} - \sum_{i=1}^{n} p_i \sum_{j=1}^{n} x_{ij} - \sum_{i=1}^{n} q_j \sum_{j=1}^{n} x_{ij}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_{ij} - \sum_{i=1}^{n} p_i (1) - \sum_{i=1}^{n} q_j (1)$$

* there is no loss in generality in assuming that the number of workers always equal to the number of jobs because we can always add fictitious workers or fictitious jobs to effect this result.

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_{ij} - \text{Constant} = z - \text{Constant.}$$

Thus the maximum or minimum value of z' and z are same. This concept is used in Hungarian method while developing an iterative method to solve the assignment problem.

3.13 Hungarian Method

Various steps involved in the computational procedure for obtaining an optimum assignment are as follows:

Step 1. Consider a square cost matrix (if it is not square make it a square matrix by applying dummy rows or columns with zero cost)

Step 2. (i) Reduce each row by lowest element in that row.

(ii) Reduce each column by lowest element in that column.

Step 3. Examine each row of the matrix.

If a row with exactly single zero is found. Mark this zero by enrectangling ($\boxed{0}$) and cross out (×) all the other zeros of the corresponding column. Proceed until all the rows have been examined. If a row with two zeros, then the choice is arbitrary.

Repeat the process for columns.

If each row and each column has one and only one marked zero, then optimum allocation is attained which is indicated by marked positions. Otherwise go to step 4

Step 4. Draw the minimum number of lines passing through all the zeros as follows:

(i) Tick (✓) rows that do not have assignments.

(ii) Tick (✓) columns that have zeros in ticked rows.

(iii) Tick (✓) rows that have assignment in ticked j columns.

(iv) Repeat (ii) and (iii) until the chain is completed.

(v) Draw straight lines through all unticked rows and ticked columns.

Step 5. If the minimum number of lines passing through all the zeros is equal to the number of rows or columns, the optimum solution is attained by an arbitrary allocation in the positions of the zeros not crossed in step 3. Otherwise go to next step.

Step 6. (i) Find the smallest element not covered by any of the lines of step 4.

(ii) Subtract this from all the uncrossed elements and add the same at the point of intersection of the two lines.

(iii) Other elements crossed by the lines remain unchanged.

Step 7. Go to step 4 and repeat the procedure till an optimum solution is attained.

Note: If a particular assignment may not be permissible, then we assign a very high cost (say ∞) for such assignment and proceed as usual.

If the assignment problem is to maximize, then convert the matrix to an opportunity loss, a matrix by subtracting each element form the highest element of the matrix. Minimization of the resulting matrix is the same as the maximization of the original effective matrix. In fact,

Consider an assignment problem

$$\text{Max } Z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}\text{ . Let } K = \text{Max } \{x_{ij}\}_{1 \le i,j \le n}\text{. Then}$$

$$Z' = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\left(K - x_{ij}\right) = K\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} - \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} = K\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} - Z = M - Z,$$

$$\text{where } M = K\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} \ge Z \text{ as } K \ge x_{ij}, \text{ for all } i,j.$$

Therefore Z' attains minimum if Z is maximum.

Example 1: Solve the assignment model shown below using Hungarian method

Job

Worker	1	2	3	4	5
1	3	8	2	10	3
2	8	7	2	9	7
3	6	4	2	7	5
4	8	4	2	3	5
5	9	10	6	9	10

Solution:

Step 1. Consider the given square cost matrix

3	8	2	10	3	1
8	7	2	9	7	1
6	4	2	7	5	1
8	4	2	3	5	1
9	10	6	9	10	1
1	1	1	1	1	5

Step 2. (i) Lowest element in the first row is 2. Subtract 2 from each element of the first row. Lowest element in the second row is 2. Subtract 2 from each element of second

row. Similarly subtract 2, 2 and 6 from third, fourth and fifth rows respectively. The above matrix reduces to

1	6	0	8	1	1
6	5	0	7	5	1
4	2	0	5	3	1
6	2	0	1	3	1
3	4	0	3	4	1
1	1	1	1	1	[5]

Similarly subtracting lowest elements 1, 2, 0, 1 and 1 of first, second, third, fourth and fifth columns respectively from their elements we get

0	4	0	7	0	1
5	3	0	6	4	1
3	0	0	4	2	1
5	0	0	0	2	1
2	2	0	2	2	1
1	1	1	1	1	[5]

Step 3. Second row has exactly single zero in column 3. Marking this zero as [0] and crossing out all other zeros in the column 3, we get

0	4	×	7	0	1
5	3	[0]	6	4	1
3	0	×	4	2	1
5	0	×	0	2	1
2	2	×	2	2	1
1	1	1	1	1	[5]

Now we observe that third row has exactly one in column 2, thus marking this enrectangling this zero and crossing all the other zeros in the column 2, we get

0	4	×	7	0	1
5	3	[0]	6	4	1
3	[0]	×	4	2	1
5	×	×	0	2	1
2	2	×	2	2	1
1	1	1	1	1	[5]

The row 4 contains only one zero in column 4, enrectangle this and discarding other zeros in the column 4.

0	4	×	7	0	1
5	3	[0]	6	4	1
3	[0]	×	4	2	1
5	×	×	[0]	2	1
2	2	×	2	2	1
1	1	1	1	1	[5]

No row with exactly one zero appear. First row contains two zeros in column 1 and column 5. Choosing arbitrarily say column 1, following the above steps by marking zero in the first column and crossing the zero in the column 5, we get

[0]	4	×	7	×	1
5	3	[0]	6	4	1
3	[0]	×	4	2	1
5	×	×	[0]	2	1
2	2	×	2	2	1
1	1	1	1	1	[5]

The row 4 has no assignment, thus optimum allocation is not attained. We go to next step.

Step 4.

(i) Tick (✓) rows that do not have assignments.

[0]	4	×	7	×	
5	3	[0]	6	4	
3	[0]	×	4	2	
5	×	×	[0]	2	
2	2	×	2	2	✓

(ii) Tick(✓)columns that have zeros (marked by ×) in ticked rows.

		✓			
[0]	4	×	7		
5	3	[0]	6	4	
3	[0]	×	4	2	
5	×	×	[0]	2	
2	2	×	2	2	✓

(iii) Tick (✓) rows that have assignment in ticked *j* columns.

		✓			
[0]	4	×	7	×	
5	3	[0]	6	4	✓
3	[0]	×	4	2	
5	×	×	[0]	2	
2	2	×	2	2	✓

Now no other row remains to repeat the ticking.

(iv) Draw minimum straight lines through all un-ticked rows and ticked columns.

		✓			
[0]	4	×	7	×	
5	3	[0]	6	4	✓
3	[0]	×	4	2	
5	×	×	[0]	2	
2	2	×	2	2	✓

Step 6. (i) The smallest element not covered by any of the lines is 2.

(ii) Subtracting 2 from all the uncrossed elements and adding the same at the point of intersection of the two lines, keeping other elements crossed by the lines remain unchanged, we get

		✓			
[0]	4	2	7	×	
3	1	[0]	4	2	✓
3	[0]	2	4	2	
5	×	2	[0]	2	
0	0	×	0	0	✓

New table is **Table – 1**

0	4	2	7	0
3	1	0	4	2
3	0	2	4	2
5	0	2	0	2
0	0	0	0	0

Step 7. Repeating the step 3 we get.

[0]	4	2	7	×
3	1	[0]	4	2
3	[0]	2	4	2
5	×	2	[0]	2
×	×	×	×	[0]

Now, since each row and each column has one and only one assignment, an optimal solution is reached. The optimal assignment is:

Worker	Assign to the job
1	1
2	3
3	2
4	4
5	5

The minimum total cost will be $z = 3 + 2 + 4 + 3 + 5 = 17.$

Example 2. A computer center has got three expert programmers. The center needs three application programs to be developed. The Head of the computer center, after studying carefully the programs to be developed, estimates the computer time in minutes required by the experts to the application programs as follows:

		Programs		
		A	*B*	*C*
Programmers	1	120	100	80
	2	80	90	110
	3	110	140	120

Assign the programmers to the programs in such a way that the total computer time is least.

Solution:

120	100	80
80	90	110
110	140	120

Subtracting the row minima and column minima from their respectively rows and columns we get

40	20	0
0	10	30
0	30	10

Choosing a zero in a row, discarding other zeros in the column and repeating the process for other columns and rows we get

40	20	[0]
[0]	10	30
×	30	10

The row 3 has no zero cells. Thus choosing the row 3 and the column 1 at which the mark × appears in the row 3. Further choose second row as it contains zero in column 1. Drawing minimum number of lines so as to cover all the rows not marked and marked columns, we get

✓			
40	20	[0]	
[0]	10	30	✓
×	30	10	✓

The minimum in the remaining cells not covered by the lines is 10. Adding 10 the cells of intersection of the lines and subtracting 10 from each of the cells not covered by the lines we get.

50	20	0
0	0	20
×	20	0

Repeating the process for this new matrix we have

50	20	[0]
×	[0]	20
[0]	20	×

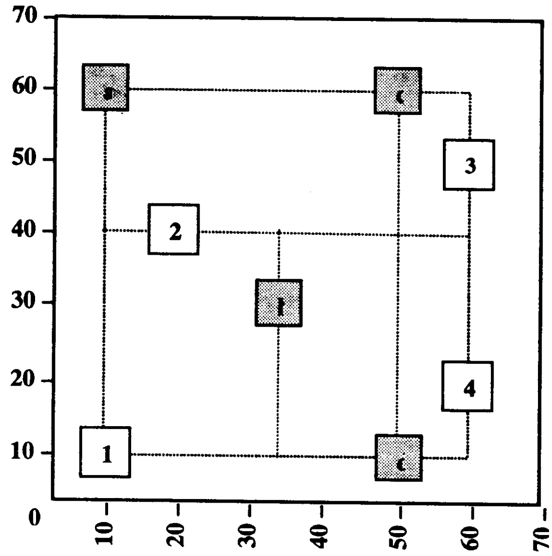
This matrix contains exactly one marked zero in each row and in each column. Hence the optimal solution is reached.

The optimal solution is: Assign programmer 1 to program 3, programmer 2 to program 2 and programmer 3 to program 1.

--	--	80
--	90	--
110	--	--

The minimum time is $z = 80 + 90 + 110 = 280$ minutes.

Example 3. The following figure gives a schematic layout of a machine shop with its existing work centers designated by squares 1, 2, 3 and 4. Four new work centers are to be added to the shop at the locations designated by circles *a*, *b*, *c* and *d*. The objective is to assign the new centers to the proposed locations in a manner that will minimize the total materials handling traffic between the existing centers and the proposed ones. Solve the problem using Hungarian method (units along the axis indicate the distance).



Solution: Given problem in tabular form is

	a	*b*	*c*	*d*
1	50	45	90	40
2	30	25	50	60
3	60	45	20	50
4	90	45	50	20

Subtracting row minima we get:

10	5	50	0
5	0	25	35
40	25	0	30
70	25	30	0

Subtracting column minima we get.

5	5	50	0
0	0	25	35
35	25	0	30
65	25	30	0

First iteration:

			✓	
5	5	50	[0]	✓
[0]	×	25	35	
35	25	[0]	30	
65	25	30	×	✓

Minimum in the cells not crossed by the lines is 5. Hence subtracting 5 from each of the cells not crossed by the lines and adding 5 to the intersections of the lines we get.

[0]	5	45	×
×	[0]	25	40
35	25	[0]	40
60	20	25	[0]

Since each row and each column contains exactly one marked zero, an optimum assignment if reached. Optimum solution is $1 \to a$, $2 \to b$, $3 \to c$ and $4 \to d$. The total minimum distance is $Z = 50 + 25 + 20 + 20 = 105$.

Example 4. The owner of a small machine shop has four machinists available to assign to jobs for the day. Five jobs are offered with expected profit for each machinist on each job as follows:

	A	*B*	*C*	*D*	*E*
1	62	78	50	101	82
2	71	84	61	73	59
3	87	92	111	71	81
4	48	64	87	77	80

Find the assignment of machinists to jobs that will result in a maximum profit. Which job should be declined?

Solution: Given AP in tabular form is :

62	78	50	101	82
71	84	61	73	59
87	92	111	71	81
48	64	87	77	80

Since the given problem concerns maximizing profit, we convert it into that of minimization by subtracting all the elements of the profit matrix from the largest element. The largest element is 111. Also we add a dummy row to balance the problem (as number of rows is not equal to the number of columns).

49	33	61	10	29
40	27	50	38	52
24	19	0	40	30
63	47	24	34	31
0	0	0	0	0

First iteration: Subtracting row minima from each row we get,

39	23	51	[0]	19
13	[0]	23	11	25
24	19	[0]	40	30
39	23	×	10	7
[0]	×	×	×	×

Drawing minimum number of lines so as to cover all marked zero we get,

		✓			
39	23	51	[0]	19	
13	[0]	23	11	25	
24	19	[0]	40	30	✓
39	23	×	10	7	✓
[0]	×	×	×	×	

Minimum in the cells not crossed by the lines is 7. Subtracting 7 from each cells not covered by the lines and adding the same to the cells of intersection of the lines we get,

39	23	58	0	19
13	0	30	11	25
17	12	0	33	23
32	16	0	3	0
0	0	7	0	0

Second iteration: Repeating the process, we get

39	23	58	[0]	19
13	[0]	30	11	25
17	12	[0]	33	23
32	16	×	3	[0]
[0]	×	7	×	×

Since each row and each column has exactly one marked zero, as optimal solution is reached. The optimal solution is $1 \to D$, $2 \to B$, $3 \to C$, $4 \to E$. Job A should be declined as it corresponds to the dummy row. The maximum profit according to this assignment schedule will be 376.

3.14 Travelling Salesman Problem

Travelling salesman problem is very similar to the assignment problem. In Travelling salesman problem (TSP), there is an additional restriction that a salesman who starts from the office should return back to the office to report by visiting all the assigned places or works. This can be achieved just by modifying the final iteration in the assignment problem. The rows and columns correspond to the places, each entry in the matrix corresponds to distance (if the object is to minimize distance), cost (if the object is to minimize cost), time (if the object is to reduce time), etc., between the places.

Example 1. Solve the following travelling salesman problem so as to minimize the cost per cycle.

From	To				
	A	*B*	*C*	*D*	*E*
A	-	3	6	2	3
B	3	-	5	2	3
C	6	5	-	6	4
D	2	2	6	-	6
E	3	3	4	6	-

Solution: Given TSP is

∞	3	6	2	3
3	∞	5	2	3
6	5	∞	6	4
2	2	6	∞	6
3	3	4	6	∞

First iteration: Subtracting row minima from each row we get,

∞	1	4	0	1
1	∞	3	0	1
2	1	∞	2	0
0	0	4	∞	4
0	0	1	3	∞

Subtracting column minima from each column and drawing minimum number of lines so as to cover all marked zero we get,

			✓		
∞	1	3	[0]	1	✓
1	∞	2	×	1	✓
2	1	∞	2	[0]	
[0]	×	3	∞	4	
×	×	[0]	3	∞	

Minimum in the cells not crossed by the lines is 1. Subtracting 1 from each cells not covered by the lines and adding the same to the cells of intersection of the lines we get,

∞	[0]	2	×	×
×	∞	1	[0]	×
2	1	∞	3	[0]
[0]	×	3	∞	4
×	×	[0]	4	∞

Since each row and each column contains exactly one marked zero, an optimal solution is reached. Optimal solution is $A \to B$, $B \to D$, $D \to A$, $C \to E$, and $E \to C$. This assignment schedule does not provide us the solution of traveling salesman problem as it gives $A \to B$, $B \to D$, $D \to A$, while D is not allowed to follow A unless C and E are processed.

Second iteration: We now find the next minimum element in the matrix (non-zero). The next minimum value is 1. So we try to bring 1 into the solution. But the element 1 occurs at two places. We shall consider all the cases separately until the acceptable solution is reached.

∞	0	2	[0]	×
×	∞	[1]	0	×
2	1	∞	3	[0]
×	[0]	3	∞	4
[0]	×	0	4	∞

We start with marking an assignment at (2, 3) instead of zero assignment at (2, 4), the resulting feasible solution then will be $A \to D, D \to B, B \to C, C \to E$ and $E \to A$.

When an assignment is made at (3, 2) instead of (3, 5), the resulting feasible solution will be $A \to E, E \to C, C \to B, B \to D$ and $D \to A$.

∞	0	2	0	[×]
×	∞	1	[0]	×
2	[1]	∞	3	0
[0]	×	3	∞	4
×	0	[0]	4	∞

Both the cases, the assignment gives a Hamiltonian circuit. The corresponding total set-up cost is 21.

Example 2. A book sales person who lives in *A* must call once a month on four customers located in *B, C, D* and *E*. The following table gives the distances in miles among the different cities.

	A	*B*	*C*	*D*	*E*
A	0	500	150	350	400
B	500	0	450	300	200
C	150	450	0	200	150
D	350	300	200	0	200
E	400	200	150	200	0

The objective is to minimize the total distance traveled by the salesperson. Solve this problem using Hungarian method.

Solution: Given problem is

	A	*B*	*C*	*D*	*E*
A	∞	500	150	350	400
B	500	∞	450	300	200
C	150	450	∞	200	150
D	350	300	200	∞	200
E	400	200	150	200	∞

After subtracting row minima and column minima, and, choosing the initial basic cells we get,

		✓		✓	
∞	300	[0]	150	250	✓
300	∞	250	50	[0]	✓
[0]	250	×	×	×	
150	50	×	∞	×	✓
250	[0]	×	×	∞	

Minimum in the cells not covered by the lines is 50, thus adding 50 to the cells of intersection of lines and subtracting from the cells not covered by the lines we get,

∞	250	0	100	250
250	∞	250	0	0
0	250	∞	0	50
100	0	0	∞	0
250	0	50	0	∞

Second iteration:
Repeating the process for the new table we get,

∞	250	[0]	100	250
250	∞	250	×	[0]
[0]	250	∞	×	50
100	[0]	×	∞	×
250	×	50	[0]	∞

Since each column and each row contains exactly one marked zero the optimal assignment is reached.

Optimal solution is: $A \to C$, $C \to A$, $B \to E$, $E \to D$, $D \to B$. Thus the assignment is not cyclic. To get a cyclic assignment we now find the next minimum (non-zero) value that is 50 in the table. There are two choices for 50 and four different possible cases to consider, they are:

Case 1: Replacing (5, 3) by (5,4): there are three sub cases: they are

Sub case 1:

∞ 0	250 500	[0] 150	100 350	250 400
250 500	∞ 0	250 450	0 300	[0] 200
[0] 150	250 450	∞ 0	0 200	50 150
100 350	[0] 300	0 200	∞ 0	0 200
250 400	0 200	50 150	[0] 200	∞ 0

Schedule is $A \to D \to B \to E \to C \to A$: cost of the schedule is 1150.

Sub case 2:

∞ 0	250 500	[0] 150	100 350	250 400
250 500	∞ 0	250 450	0 300	[0] 200
[0] 150	250 450	∞ 0	0 200	50 150
100 350	[0] 300	0 200	∞ 0	0 200
250 400	0 200	50 150	[0] 200	∞ 0

Schedule is: $A \to B \to D \to E \to C \to A$: cost of the schedule is 1300

Sub case 3:

∞ 0	250 500	[0] 150	100 350	250 400
250 500	∞ 0	250 450	0 300	[0] 200
[0] 150	250 450	∞ 0	0 200	50 150
100 350	[0] 300	0 200	∞ 0	0 200
250 400	0 200	50 150	[0] 200	∞ 0

Schedule is $A \to B \to E \to C \to D \to A$: cost of the schedule is 1400

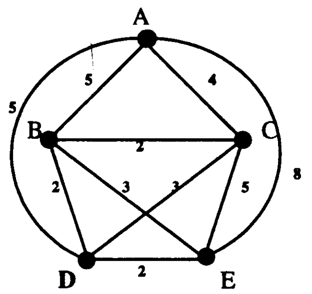
Case 2: Replacing (3, 1) by (3, 5)

∞ (0)	250 (500)	0 (150)	100 (350)	250 (400)
250 (500)	∞ (0)	250 (450)	0 (300)	0 (200)
0 (150)	250 (450)	∞ (0)	0 (200)	50 (150)
100 (350)	0 (300)	0 (200)	∞ (0)	0 (200)
250 (400)	0 (200)	50 (150)	0 (200)	∞ (0)

Schedule is *A→C→E→B→D→A*: cost of the schedule is 1150 (this is just a reverse Hamiltonian cycle obtained in sub case 1 of case 1).

Conclusion: The optimum value of the schedule is thus 1150, and the optimum solution is the schedule is A→C→E→B→D→A (or reverse).

Example 3: Find the minimal Hamiltonian circuit of the following graph using Hungarian method



Solution: Given problem in matrix form is

	A	B	C	D	E
A	∞	5	4	5	8
B	5	∞	2	2	3
C	4	2	∞	3	5
D	5	2	3	∞	2
E	8	3	5	2	∞